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Complexity of Weighted Approximation over $\mathbb{R}^1$

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We study approximation of univariate functions defined over the reals. We assume that the rth derivative of a function is bounded in a weighted L_p norm with a weight ψ. Approximation algorithms use the values of a function and its derivatives up to order $r-1$. The worst case error of an algorithm is defined in a weighted L_q norm with a weight ρ. We study the worst case (information) complexity of the weighted approximation problem, which is equal to the minimal number of function and derivative evaluations needed to obtain error ε. We provide necessary and sufficient conditions in terms of the weights ψ and ρ, as well as the parameters r, p, and q for the weighted approximation problem to have finite complexity. We also provide conditions which guarantee that the complexity of weighted approximation is of the same order as the complexity of the classical approximation problem over a finite interval. Such necessary and sufficient conditions are also provided for a weighted integration problem since its complexity is equivalent to the complexity of the weighted approximation problem for $q=1$. © 2000 Academic Press

1. INTRODUCTION

To motivate the setting of this paper we begin with a classical approximation problem, which is defined as the approximation of smooth univariate functions over a bounded domain. The smoothness of functions means the

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existence of rth derivatives whose L_p norms are uniformly bounded. The bounded domain can be given as an interval $[-R, R]$ for some finite R. We approximate the function by algorithms that use function and derivative values at some sample points from the domain $[-R, R]$. The worst case error is measured in the L_q norm, and the worst case (information) complexity comp(ε, R) is equal to the minimal number of function and derivative evaluations needed to obtain error ε. We stress that the parameters p and q are not related. Let

$$\gamma = r + 1/q - 1/p$$

be positive, and let $s = \gamma$ for $p \leqslant q$, and $s = r$ for $p > q$. It is known, see, e.g., [5, 6], that

$$\mathrm{comp}(\varepsilon, R) = \Theta\left(\left(\frac{R^\gamma}{\varepsilon}\right)^{1/s}\right)$$

with the factors in the Theta notation[2] independent of ε and R. Hence the complexity goes to infinity with R.

In this paper, we study approximation of smooth univariate functions defined over the unbounded domain of reals, $\mathbb{R} = (-\infty, +\infty)$. To obtain positive results we modify the classical approximation problem by introducing weight functions ψ and ρ. The function ψ is a weight of the L_p norm that is used for bounding the rth derivatives. The function ρ is a weight of the L_q norm that defines the error of algorithms. We make a few natural assumptions about the weights ψ and ρ: they are nonnegative, positive at a neighborhood of zero, and even. We study the worst case complexity of this weighted approximation problem, which is proportional to the minimal number of function and derivative evaluations needed to guarantee error ε.

We now motivate the weighted approximation problem for univariate functions over the reals. Many practical problems are defined over the reals. Usually it is possible to reduce the problem to a finite interval by a change of variables. Unfortunately, this approach may cause singularities in the transformed function, and it is not clear how to cope with these singularities. We prefer to deal with the original problem over the infinite domain and to see the role of weights and smoothness in determining the complexity of the weighted approximation problem.

In this paper we only study univariate functions. Of course, the case of multivariate functions is much more interesting. We treat this paper as a first step towards the weighted approximation problem for multivariate

[2] Recall that $g(\varepsilon) = O(h(\varepsilon))$ means that there exist positive numbers C and ε_0 such that $g(\varepsilon) \leqslant Ch(\varepsilon)$ for $\varepsilon \in [0, \varepsilon_0]$. Then $g(\varepsilon) = \Theta(h(\varepsilon))$ means that both $g(\varepsilon) = O(h(\varepsilon))$ and $h(\varepsilon) = O(g(\varepsilon))$ hold, and $g(\varepsilon) = \Omega(h(\varepsilon))$ means that $h(\varepsilon) = O(g(\varepsilon))$.

functions. It is clear that our results for univariate functions can be directly applied for spaces of multivariate functions that are tensor products of spaces of univariate functions. Then, Smolyak's or weighted tensor product algorithms can be used, since these algorithms are built on efficient algorithms for the univariate case, see, e.g., [9, 10]. Hence, understanding of the univariate case is crucial. The isotropic case for multivariate functions is more difficult, since it cannot be decomposed into a number of univariate cases. We started the study of the isotropic case for monotonic weights and it will be reported in a future paper. The case of general weights seems much more difficult for the multivariate case.

In this paper we address two questions. The first is to find under what conditions on the weights we have finite complexity for the weighted approximation problem. It is clear that the behavior of the weights at infinity determines whether the complexity of weighted approximation is finite. We obtain a necessary and sufficient condition on the weights for weighted approximation to have finite complexity.

It is easy to see that the weighted approximation problem is not easier than classical approximation, i.e., the complexity of weighted approximation is always bounded from below by a multiple of $\varepsilon^{-1/s}$. This leads us to the second question of when the complexity of weighted approximation is of the same order in ε^{-1} as the complexity of the classical approximation problem. We first consider monotonic weights and present an algorithm that solves the weighted approximation problem with cost proportional to $\varepsilon^{-1/s}$, the complexity of the classical approximation problem. We then discuss necessary and sufficient conditions on the weights for these complexities to be equivalent.

We, now explain our results for a simplified choice of weights. Define for any function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ the *order* of g at infinity as

$$o_g = \sup\{\beta \in \mathbb{R} : \lim_{t \to \infty} g(t)\, t^{\beta} = 0\}, \tag{1}$$

with $o_g = -\infty$ if the corresponding set is empty. Observe that if $g(t) = \Theta(t^k)$ as t goes to infinity then $o_g = -k$, and for the function $1/g(t)$ we have $o_{1/g} = k$. On the other hand, for $g(t) = \exp(-t)$ we have $o_g = \infty$ and $o_{1/g} = -\infty$. In general, $o_g \leqslant -o_{1/g}$ and there exist functions g for which $o_g \neq -o_{1/g}$.

We now make the simplifying assumption that $o_\rho = -o_{1/\rho}$ and $o_\psi = -o_{1/\psi}$, both being finite numbers. Then the complexity of weighted approximation is of the *same order* as the complexity of the classical approximation problem if

$$o_\rho + \min\{1 - 1/p, o_{1/\psi}\} > \gamma.$$

The complexity of weighted approximation is *infinite* for small ε if

$$o_\rho + \min\{1 - 1/p, o_{1/\psi}\} < \gamma.$$

Finally, if $o_\rho + \min\{1 - 1/p, o_{1/\psi}\} = \gamma$ then *anything* can happen, i.e., the complexity of weighted approximation may be of the same order or much larger than the complexity of the classical approximation problem, or can be infinity depending on the specific form of the weights. We provide examples of all three possibilities and pose one conjecture at the end of the paper.

For weights with infinite order, the complexity of weighted approximation can be also either finite or infinite. In particular, for $\psi(t) = \rho(t) = \exp(-|t|)$ we have infinite complexity, whereas for $\psi(t) = \rho(t) = \exp(-t^2)$ the complexity is finite.

Formally, this paper is devoted only to the weighted approximation problem. However, as explained in Remark 2, the results for $q = 1$ are relevant for a weighted integration problem where, instead of approximating functions, one is interested in approximating integrals $S(f) = \int_{\mathbb{R}} f(x)\, \rho(x)\, dx$. Classically, see, e.g., [2], such weighted integrals are approximated using either a change of variables or Gaussian quadrature. As with weighted approximation, the drawback of the change of variables may be the artificial introduction of singularities. Gaussian quadratures are, in general, difficult to derive and need not be optimal for functions with finite regularity as assumed in this paper.

There are only a few optimality results dealing with weighted integration problems for functions of a finite regularity r, as assumed in this paper. Among them are [1, 4] where $r = 1$, $\psi \equiv 1$ and ρ is the density of a Gaussian distribution. For the weighted approximation problem, optimal algorithms and optimal information were considered in Appendix G of [7] for $r = 1$, $p = q = \infty$, and $\psi \equiv 1$. This paper provides optimal algorithms for various regularities ($r \geqslant 1$) and weights ρ and ψ. These algorithms have already been implemented and tested for some weight functions and some values of r; see [3].

2. WEIGHTED APPROXIMATION

In this section we define a weighted approximation problem. To motivate our definition we first consider a classical approximation problem over bounded domains. For the interval $B_R = [-R, R]$, consider the following class of functions

$$\mathscr{W}^r_p(B_R) = \{f\colon B_R \to \mathbb{R} : f^{(r-1)} \text{ is absolutely continuous and } \|f^{(r)}\|_p < \infty\}.$$

By $\|f\|_p$ we denote the norm in the space $L_p(B_R)$. Here, r is a positive integer, and p can be any number from $[1, \infty]$. Recall that for $p=\infty$, $\|g\|_{L_\infty} = \sup \mathrm{ess}_{x \in B_R} |g(x)|$. Consider the unit semi-ball $\mathcal{F}$ of functions from the space $\mathcal{W}_p^r(B_R)$:

$$\mathcal{F} = \{ f \in \mathcal{W}_p^r(B_R) : \|f^{(r)}\|_p \leqslant 1 \}.$$

We approximate functions f from the class $\mathcal{F}$. The error of an approximation is measured in the norm of the space $L_q(B_R)$, where q can be any number from $[1, \infty]$. That is, given a positive error demand ε, where $\varepsilon \leqslant 1$, we wish to have an algorithm $\mathcal{U}$ with the error $e(\mathcal{U})$ not exceeding ε. The error $e(\mathcal{U})$ is defined by

$$e(\mathcal{U}) = \sup_{f \in \mathcal{F}} \|f - \mathcal{U}(f)\|_q.$$

We assume that we can compute function values $f(t)$ and/or derivative values $f^{(j)}(t)$ for $t \in B_R$, and that $\mathcal{U}(f)$ is of the form

$$\mathcal{U}(f) = \phi(\mathcal{N}(f)) \qquad \text{with} \quad \mathcal{N}(f) = [f^{(j_1)}(x_1), ..., f^{(j_n)}(x_n)]$$

for some mapping ϕ, points $x_i \in B_R$ that are not necessarily distinct, and $j_i \leqslant r-1$. The parameter n is the cardinality of information $\mathcal{N}$, which we will refer to by $\mathrm{card}(\mathcal{N})$.[3]

It is intuitively clear that the approximation problem depends, in particular, on the radius R and that it is harder for larger R. Indeed, let $r(n, R)$ denote the minimal error among all possible $\mathcal{U}$ which use information of cardinality n. It is well known (see, e.g., [8]) that, modulo a multiplicative factor $c \in [1, 2]$, we have

$$r(n, R) = c \inf_{\mathrm{card}(\mathcal{N}) = n} \sup \{ \|f\|_q : f \in \mathcal{F}, \mathcal{N}(f) = 0 \}. \tag{2}$$

By a standard change of variables one can verify that

$$r(n, R) = R^\gamma r(n, 1) \qquad \text{with} \quad \gamma = r + 1/q - 1/p. \tag{3}$$

Moreover, there are many results establishing the behavior of $r(n, 1)$, see, e.g., [5, 6]. It equals infinity when $n \leqslant r-1$ and, for $n \geqslant r$, it is proportional to

$$r(n, 1) = \Theta(n^{-s}) \qquad \text{with} \quad s = \begin{cases} \gamma & \text{if } p \leqslant q, \\ r & \text{if } p > q. \end{cases} \tag{4}$$

[3] For the simplicity of presentation, we assume in this paper that the information $\mathcal{N}$ used by $\mathcal{U}$ is non-adaptive and of fixed cardinality. This is without any loss of generality since, as it is well known (see, e.g., [8]), adaption and varying cardinality do not help.

Observe that $s \geqslant 0$, and $s = 0$ iff $\gamma = 0$. The latter holds iff $r = 1$, $p = 1$, $q = \infty$. We need to guarantee that $r(n, 1)$ tends to zero as n goes to infinity. Therefore we assume throughout the paper that

$$\gamma = r + 1/q - 1/p > 0. \tag{5}$$

It is also known that relatively simple algorithms are almost optimal. Indeed, let $\mathscr{U}_n^*(f)$ be a piecewise polynomial of degree $r-1$ interpolating f at equally spaced points $x_{i,n} = (i-1)/(n-1)$, $i = 1, ..., n$. Then for $R = 1$

$$e(\mathscr{U}_n^*) \leqslant A_1 (n - r + 1)^{-s}, \qquad \forall n \geqslant r. \tag{6}$$

The constant A_1 depends on r, p, q but, of course, is independent of n. Moreover $A_1 \leqslant 1$. This means that modulo a multiplicative factor, the algorithm $\mathscr{U}_n^*$ has minimal error.

Remark 1. The interval $[-R, R]$ is only chosen for simplicity. For the approximation problem defined over functions $f: [a, b] \to \mathbb{R}$ the same results hold true. For instance (3) holds with $R = (b-a)/2$, and the error of the corresponding algorithm $\mathscr{U}_n^*$ for the interval $[a, b]$ satisfies

$$e(\mathscr{U}_n^*) \leqslant A_1 \left(\frac{b-a}{2} \right)^{\gamma} (n - r + 1)^{-s}, \qquad \forall n \geqslant r. \tag{7}$$

The algorithm $\mathscr{U}_n^*$ can be also used for functions f outside the semi-ball $\mathscr{F}$. Indeed, since $\mathscr{U}_n^*$ is a linear operator, $\mathscr{U}_n^*(f)$ is well defined as long as $f^{(r)} \in L_p([-R, R])$. Moreover, $\|f - \mathscr{U}_n^*(f)\|_q \leqslant e(\mathscr{U}_n^*) \cdot \|f^{(r)}\|_p$.

Let comp(ε, R) denote the minimal number of function values needed to construct $\mathscr{U}$ with error at most ε. For $R = 1$, we denote comp$(\varepsilon, 1)$ simply by comp(ε). The quantity comp(ε, R) is called the *information complexity*, for brevity the complexity,[4] of the approximation problem. From (3) we obtain

$$\mathrm{comp}(\varepsilon, R) = \mathrm{comp}(\varepsilon / R^{\gamma}).$$

From (4) and (5) we have $s > 0$ and therefore

$$\mathrm{comp}(\varepsilon) = \Theta(\varepsilon^{-1/s}). \tag{8}$$

This means that for all finite R, the complexity is finite. However, for any fixed ε, comp(ε, R) approaches infinity with R and, hence, the problem cannot be solved for $R = \infty$.

[4] Information complexity is a lower bound on the total complexity. The latter is defined as the minimal cost needed for computing an ε-approximation. In many cases, information complexity is a sharp bound on the total complexity; see [8]. This is also the case for monotonic weights as shown in Subsection 4.1.

This discussion shows that the approximation problem over the whole space $\mathbb{R}$ must be modified. Such a modification can be provided by a *weighted* approximation problem, which is defined as follows. Let

$$\psi\colon \mathbb{R} \to \mathbb{R}$$

be a nonnegative and (Lebesgue) measurable function. We call ψ a *weight* function. The regularity of functions f is defined in a weighted sense. That is, we consider

$$\mathscr{F}_p = \{ f\colon \mathbb{R} \to \mathbb{R} : f^{(r-1)} \text{ is absolutely continuous and } \|f^{(r)}\|_{p,\psi} < 1\}$$

with

$$\|f^{(r)}\|_{p,\psi} = \left\{ \int_{\mathbb{R}} |f^{(r)}(t)|^p \, \psi^p(t) \, dt \right\}^{1/p}.$$

The *weighted* approximation problem over the class $\mathscr{F}_p$ is defined as an approximation of f by $\mathscr{U}(f)$ with the modified error

$$e(\mathscr{U}) = \sup_{f \in \mathscr{F}_p} \left\{ \int_{\mathbb{R}} |f(x) - \mathscr{U}(f)(x)|^q \, \rho^q(x) \, dx \right\}^{1/q},$$

where ρ is another (nonnegative measurable) weight function. Of course, for $q = \infty$ we have

$$e(\mathscr{U}) = \sup_{f \in \mathscr{F}_p} \ \operatorname*{sup\,ess}_{x \in \mathbb{R}} |f(x) - \mathscr{U}(f)(x)| \, \rho(x).$$

For example, letting $\rho(t) = \psi(t) = 1$, if $|t| \leqslant R$, and $\rho(t) = \psi(t) = 0$ if $|t| > R$, this weighted approximation problem reduces to the classical approximation problem over the interval $[-R, R]$.

The error of an algorithm $\mathscr{U}$ now depends on both weights ρ and ψ, as well as on the parameters p and q. To stress this dependence, instead of $e(\mathscr{U})$ we will sometimes write

$$e(\mathscr{U}, \rho, \psi) \qquad \text{or} \qquad e(\mathscr{U}, \rho, \psi, q, p).$$

We add in passing that $e(\mathscr{U})$ is finite only if $\mathscr{U}$ is exact on polynomials of degree $r-1$ since they belong to the kernel of $\mathscr{F}_p$.

Let $\operatorname{comp}(\varepsilon, \rho, \psi) = \operatorname{comp}(\varepsilon, \rho, \psi, p, q)$ denote the minimal number of function and derivative values needed to solve the weighted approximation problem with error at most ε. As before, we call $\operatorname{comp}(\varepsilon, \rho, \psi)$ the (information) complexity of the weighted approximation problem.

It is clear that we must impose some conditions on the weights ρ and ψ to guarantee that the complexity is finite. For example, assume that ρ is bounded and has a finite interval B as its support, and ψ is bounded from below by a positive number over B. Then the complexity is finite since the weighted approximation problem reduces to the approximation problem over a finite interval. On the other hand, we shall see that the complexity is infinite if the function $h(t) = t^{r-1}\rho(t)$ does not belong to $L_q(\mathbb{R})$, regardless of the function ψ.

We will make a number of assumptions concerning the weights ρ and ψ. Most of these assumptions are needed only to simplify the analysis and/or to exclude trivial cases for which the complexity is infinite.

Since the most interesting case is when the weights have unbounded support, we assume that, at least, ψ is always positive. We also assume that ρ and ψ are even. This is also done for simplicity only since we do not want to distinguish arguments that differ by a sign. To exclude trivial cases, we assume that both weight functions are positive and continuous at zero. Our last assumption is that for any finite R, we have

$$\sup \operatorname{ess}\{\rho(t) \colon t \in [0, R]\} < \infty, \tag{9}$$

$$\inf \operatorname{ess}\{\psi(t) \colon t \in [0, R]\} > 0. \tag{10}$$

That is, we assume that ρ is bounded from above by a finite number and ψ is bounded from below by a positive number over finite intervals. Of course, these assumptions are satisfied by monotonic weight functions.

In summary, we make the following assumptions on the weights:

Assumption 1. The weight ρ is nonnegative, measurable, even, positive, and continuous at zero, and satisfies (9) for all nonnegative R.

Assumption 2. The weight ψ is positive, measurable, even, continuous at zero, and satisfies (10) for all nonnegative R.

For such weights ρ and ψ, we study the following two problems:

Problem 1. When is the complexity of weighted approximation finite for every nonzero ε?

Problem 2. When is the complexity of weighted approximation of the same order as $\operatorname{comp}(\varepsilon)$?

Since ρ and ψ are positive and continuous at zero, they do not vanish around zero. Therefore, the weighted approximation problem is not easier than the approximation problem over $[-R, R]$ for some positive R, i.e., $\operatorname{comp}(\varepsilon, \rho, \psi) = \Omega(\operatorname{comp}(\varepsilon))$. Thus, the equivalence of complexities addressed in question (2) holds iff $\operatorname{comp}(\varepsilon, \rho, \psi) = O(\operatorname{comp}(\varepsilon))$.

We end this section with the following remark concerning weighted integration and its relation to weighted approximation.

Remark 2. To simplify the presentation, the paper deals only with weighted *approximation* problems. We want to stress, however, that all results of this paper are also valid for the following *weighted integration* problem. Let p, ρ, and ψ be as before. For any function f from the class $\mathscr{F}_p$, we want to approximate the weighted integral

$$\mathrm{Int}_\rho(f) = \int_{\mathbb{R}} f(x)\, \rho(x)\, dx.$$

It is easy to show that this weighted integration problem is equivalent to the weighted approximation problem with the same weights, the same value of p, and with $q = 1$. That is, if $\mathrm{comp}(\varepsilon, \mathrm{Int}_\rho)$ denotes the minimal number of function evaluations needed to approximate Int_ρ with the error not exceeding ε, then

$$\mathrm{comp}(\varepsilon, \mathrm{Int}_p) = \Theta(\mathrm{comp}(\varepsilon, \rho, \psi, 1, p)).$$

Moreover, for any algorithm $\mathscr{U}^{app}$ for the weighted approximation problem with $q = 1$,

$$\mathscr{U}^{int}(f) = \int_{\mathbb{R}} \mathscr{U}^{app}(f)\, \rho(x)\, dx$$

is a quadrature with the error proportional to the error of $\mathscr{U}^{app}$. In particular, $\mathscr{U}^{int}$ is almost optimal if $\mathscr{U}^{app}$ is almost optimal. Thus, this paper provides answers to the above 2 questions also for weighted integration.

3. FINITE COMPLEXITY

In this section we provide a necessary and sufficient condition for the complexity of the weighted approximation problem to be finite. The condition is expressed in terms of the nonlinear functional L defined by

$$L(R) = \sup_{\|\alpha\|_p \leqslant 1} \left(\int_R^\infty \rho^q(x) \left(\int_R^x \frac{(x-t)^{r-1}}{(r-1)!} \frac{|\alpha(t)|}{\psi(t)}\, dt \right)^q dx \right)^{1/q}, \qquad \forall R \in \mathbb{R}_+ \tag{11}$$

Observe that the integral over $[R, x]$ is well-defined since ψ is measurable and bounded away from zero. Therefore $L(R)$ is also well-defined, although it may happen that $L(R) = \infty$. Note also that the supremum in (11) is attained by functions α whose support is contained in $[R, \mathscr{U})$. Moreover, L is nonincreasing. By a change of variables in both integrals, (11) can be rewritten as

$$L(R) = R^{\gamma} \sup_{\|\alpha\|_p \leqslant 1} \left(\int_1^{\infty} \rho^q(Rx) \left(\int_1^x \frac{(x-t)^{r-1}}{(r-1)!} \frac{|\alpha(t)|}{\psi(Rt)} \, dt \right)^q dx \right)^{1/q}, \tag{12}$$

where γ is given by (5).

The nonlinear functional L controls the behavior of functions from the semi-ball $\mathscr{F}_p$ over the interval $[R, \infty)$ and (by symmetry) over $(-\infty, -R]$. Indeed, $L(R)$ is the weighted norm of the error of the Taylor polynomial which approximates the function using $f^{(i)}(R)$ for $i = 0, 1, ..., r-1$. Since we can sample the function f only finitely many times, it is clear that for large R, the set $A = (-\infty, -R] \cup [R, \infty)$ does not contain sample points and the behavior of the functions over A is controlled only by a priori information given by the parameters r, p, q and the weights ρ and ψ. It is therefore natural to expect that the error of approximation can be arbitrarily small only if $L(R)$ goes to zero as R tends to infinity. The formal proof is given below.

THEOREM 1. *The complexity* $\operatorname{comp}(\varepsilon, \rho, \psi)$ *is finite for every* $\varepsilon > 0$ *iff*

$$\lim_{R \to \infty} L(R) = 0. \tag{13}$$

Proof. Suppose first that the complexity is finite for all positive ε. Then for any positive ε there exists an algorithm $\mathscr{U}_n$ using information $\mathscr{N}_n(f)$ that consists of function/derivative evaluations at points $t_{1,n}, ..., t_{n,n}$ whose error $e(\mathscr{U}_n, \rho, \psi)$ is at most ε. Here, $n = n(\varepsilon)$ is an integer. Let

$$R_{\varepsilon} = \max_{1 \leqslant i \leqslant n} |t_{i,n}|.$$

For $R \geqslant R_{\varepsilon}$ take an arbitrary function α with $(\int_R^{\infty} |\alpha(t)|^p \, dt)^{1/p} \leqslant 1$. Let $f_{\varepsilon, \alpha}$ be the function that vanishes on $(-\infty, R]$ and satisfies

$$f_{\varepsilon, \alpha}(x) = \int_R^x \frac{(x-t)^{r-1}}{(r-1)!} \frac{|\alpha(t)|}{\psi(t)} \, dt \qquad \text{when} \quad x \geqslant R.$$

Since $f_{\varepsilon, \alpha} \in \mathscr{F}_p$ and $\mathscr{N}_n(f_{\varepsilon, \alpha}) = 0$, we have due to (2)

$$\left(\int_{\mathbb{R}} \rho^q(x) \, |f_{\varepsilon, \alpha}(x)|^q \, dx \right)^{1/q} = \left(\int_R^{\infty} \rho^q(x) \left(\int_R^x \frac{(x-t)^{r-1}}{(r-1)!} \frac{|\alpha(t)|}{\psi(t)} \, dt \right)^q dx \right)^{1/q}$$
$$\leqslant e(\mathscr{U}_n, \rho, \psi).$$

Since α is an arbitrary function, this implies that $L(R) \leqslant e(\mathcal{U}_n, \rho, \psi) \leqslant \varepsilon$ for any $R \geqslant R_\varepsilon$. This yields (13).

Suppose now that (13) holds. Given ε, let $R = R(\varepsilon)$ be a positive number for which $L(R) \leqslant \varepsilon/3^{1/q}$. Consider now the following algorithm $\mathcal{U}_\varepsilon(f)$. We first define this algorithm for $|x| \geqslant R$. For $x \geqslant R$, it equals the Taylor polynomial at R, i.e., $\mathcal{U}_\varepsilon(f) = \sum_{i=0}^{r-1} f^{(i)}(R)(x-R)^i/i!$. For $x \leqslant -R$, it is also given by a Taylor polynomial, this time at $-R$. For $x \geqslant R$, we have

$$f(x) - \mathcal{U}_\varepsilon(x) = \int_R^x \frac{(x-t)^{r-1}}{(r-1)!} \frac{\alpha(t)}{\psi(t)}\, dt \qquad \text{with} \quad \alpha(t) = f^{(r)}(t)\, \psi(t).$$

Since $(\int_R^\infty |\alpha(t)|^p\, dt)^{1/p} \leqslant \|\alpha\|_p \leqslant 1$, we conclude that

$$\left(\int_R^\infty |f(x) - \mathcal{U}_\varepsilon(f)(x)|^q\, \rho^q(x)\, dx \right)^{1/q} \leqslant L(R) \leqslant \frac{\varepsilon}{3^{1/q}}. \tag{14}$$

The same holds for $x \leqslant -R$. Hence, the total error $e(\mathcal{U}_\varepsilon, \rho, \psi)$ will be at most ε if we define $\mathcal{U}_\varepsilon(f)(x)$ for $x \in [-R, R]$ such that

$$\left(\int_{-R}^R |f(x) - \mathcal{U}_\varepsilon(f)(x)|^q\, \rho^q(x)\, dx \right)^{1/q} \leqslant \frac{\varepsilon}{3^{1/q}}. \tag{15}$$

We now show that such a $\mathcal{U}_\varepsilon$ exists and uses finitely many function/derivative evaluations. Let $M_R = \sup\{\rho(t) \colon t \in [0, R]\}$. Due to (9), M_R is finite. To satisfy (15), it is enough to guarantee that

$$\left(\int_{-R}^R |f(x) - \mathcal{U}_\varepsilon(f)(x)|^q\, dx \right)^{1/q} \leqslant \frac{\varepsilon}{3^{1/q} M_R}.$$

Let $K_R = \inf\{\psi(t) \colon t \in [0, R]\}$. Due to (10), K_R is positive. For $f \in \mathcal{F}_p$ we thus have

$$\left(\int_{-R}^R |f^{(r)}(x)|^p\, dx \right)^{1/p} \leqslant \frac{1}{K_R} \left(\int_{-R}^R |f^{(r)}(x)|^p\, \psi^p(x)\, dx \right)^{1/p} \leqslant \frac{1}{K_R}.$$

Hence, we reduce the problem to the classical (unweighted) approximation over the finite interval $[-R, R]$. Since this problem has a finite complexity, there exists $\mathcal{U}_\varepsilon$ satisfying (15) and using finitely many function/derivatives evaluations. This completes the proof. ∎

We now elaborate on the condition (13) by presenting a more explicit form of $L(R)$. We can eliminate functions α in (11) for special values of p and q.

Case 1. Let $p=\infty$. The corresponding supremum in (11) is attained for $\alpha(t)\equiv 1$. Hence (13) takes the simplified form

$$\lim_{R\to\infty}\left(\int_R^\infty \rho^q(x)\left(\int_R^x \frac{(x-t)^{r-1}}{\psi(t)}\,dt\right)^q dx\right)^{1/q}=0.$$

For $q=\infty$, this simplifies even more to

$$\lim_{R\to\infty}\ \sup_{x\geqslant R}\text{ess}\ \rho(x)\int_R^x \frac{(x-t)^{r-1}}{\psi(t)}\,dt=0.$$

The last condition for $\psi\equiv 1$ is equivalent to

$$\lim_{R\to\infty}\rho(R)\,R^r=0.$$

When $\psi\equiv 1$ but $q<\infty$ then (13) becomes

$$\lim_{R\to\infty}\left(\int_R^\infty \rho^q(x)(x-R)^{rq}\,dx\right)^{1/q}=0.$$

Case 2. Let $q=1$. Then (13) is equivalent to

$$\lim_{R\to\infty}\ \sup_{\|\alpha\|_p\leqslant 1}\int_R^\infty \frac{|\alpha(t)|}{\psi(t)}\int_t^\infty (x-t)^{r-1}\,\rho(x)\,dx\,dt=0.$$

A standard use of Hölder's inequality allows us to eliminate α, and (13) is equivalent to

$$\lim_{R\to\infty}\left(\int_R^\infty\left(\frac{1}{\psi(t)}\int_t^\infty (x-t)^{r-1}\,\rho(x)\,dx\right)^{p/(p-1)} dt\right)^{(p-1)/p}=0.$$

Hence, regardless of the weight ψ, $\int_{\mathbb{R}} x^{r-1}\rho(x)\,dx<\infty$ is a necessary condition for the complexity to be finite.

For $q=1$ we observed that the weight ρ must go sufficiently quickly to zero (independently of the weight ψ) to make the complexity finite. We now show that the corresponding property holds for any q.

Corollary 1. *Let $h(x)=x^{r-1}\rho(x)$. If* $\text{comp}(\varepsilon,\rho,\psi)$ *is finite for every positive ε then*

$$h\in L_q(\mathbb{R}_+)\quad \textit{for}\ q<\infty \qquad \textit{and}\qquad \lim_{x\to\infty}h(x)=0\quad \textit{for}\ q=\infty.$$

Proof. We rewrite (11) as

$$L(R) = \sup_{\|\alpha\|_p \leqslant 1} \left(\int_R^\infty h^q(x) \left(\int_R^x \frac{(1-t/x)^{r-1}}{(r-1)!} \frac{|\alpha(t)|}{\psi(t)}\, dt \right)^q dx \right)^{1/q}.$$

Observe that the inner integral $\int_R^x (1-t/x)^{r-1}\, |\alpha(t)|/\psi(t)\, dt$ increases with x for any function α. Therefore, for $\lim_{R\to\infty} L(R) = 0$ it is necessary that h belongs to $L_q(\mathbb{R})$ for finite q and that $\lim_{x\to\infty} h(x) = 0$ for $q = \infty$. ∎

For arbitrary values of p and q, it is difficult to eliminate α from (13). By using Hölder's inequality we may estimate $L(R)$ by

$$L(R) \leqslant \tilde{L}(R)$$

$$:= \left(\int_R^\infty \rho^q(x) \left(\int_R^x \left(\frac{(x-t)^{r-1}}{(r-1)!\, \psi(t)} \right)^{p/(p-1)} dt \right)^{q(p-1)/p} dx \right)^{1/q} \qquad (16)$$

and use the convergence of $\tilde{L}(R)$ to zero as a sufficient condition for finite complexity.

We now translate the condition (13) in terms of the behavior of the weight functions at infinity. This will be done by using the order at infinity defined in (1).

THEOREM 2. (i) *Assume that the set* $\{o_\rho, o_{1/\psi}\}$ *is different from the set* $\{\infty, -\infty\}$. *Then*

$$o_\rho + \min\{1-1/p,\, o_{1/\psi}\} > \gamma = r + 1/q - 1/p \qquad (17)$$

implies finite complexity for all positive ε.

(ii) *Assume that the set* $\{o_{1/\rho}, o_\psi\}$ *is different from the set* $\{\infty, -\infty\}$. *Then*

$$-o_{1/\rho} + \min\{1-1/p,\, -o_\psi\} < \gamma = r + 1/q - 1/p \qquad (18)$$

implies infinite complexity for small positive ε.

Proof. We assume that both p and q are from $(1, \infty)$. The limiting cases can be shown by a slight modification of the proof presented below.

To prove (i), observe that $o_\rho + \min\{1-1/p, o_{1/\psi}\}$ is well-defined since the case $\infty - \infty$ is excluded. We may also assume that both o_ρ and $o_{1/\psi}$ are finite. For any positive δ, we have

$$\rho(x) = O(x^{-o_\rho+\delta}) \qquad \text{and} \qquad 1/\psi(x) = O(x^{-o_{1/\psi}+\delta}).$$

Consider the integral

$$a := \int_R^x \left(\frac{(x-t)^{r-1}}{(r-1)!\,\psi(t)} \right)^{p/(p-1)} dt$$
$$= O\left(x^{(r-1)\,p/(p-1)} \int_R^x t^{(-o_{1/\psi}+\delta)\,p/(p-1)}\, dt \right).$$

We can always choose δ such that $(-o_{1/\psi}+\delta)\,p/(p-1) \neq -1$. Therefore

$$a = O\left(x^{(r-1)\,p/(p-1)}\, \frac{x^{1+(-o_{1/\psi}+\delta)\,p/(p-1)} - R^{1+(-o_{1/\psi}+\delta)\,p/(p-1)}}{1+(-o_{1/\psi}+\delta)\,p/(p-1)} \right)$$
$$= O(x^{(r-1+\max\{0,\,-o_{1/\psi}+\delta+1-1/p\})\,p/(p-1)}).$$

Then the integrand of (16) can be estimated as

$$b := \rho^q(x) \left(\int_R^x \left(\frac{(x-t)^{r-1}}{(r-1)!\,\psi(t)} \right)^{p/(p-1)} dt \right)^{q(p-1)/p}$$
$$= O(x^{-q(o_\rho-\delta-r+1-\max\{0,\,-o_{1/\psi}+\delta+1-1/p\})}).$$

Since $-\max\{0, \alpha\} = \min\{0, -\alpha\}$ for any α, we have

$$b = O(x^{-q(o_\rho+\min\{0,\,o_{1/\psi}-\delta-1+1/p\}-r+1-\delta)}).$$

This function is integrable since $q(o_\rho+\min\{0, o_{1/\psi}-\delta-1+1/p\}-r+1-\delta)>1$ for sufficiently small δ. Hence, $\tilde{L}(R)$ goes to zero as R goes to infinity. This implies that the complexity is finite.

To prove (ii), observe that $-o_{1/\rho}+\min\{1-1/p, -o_\psi\}$ is well-defined since the case $\infty-\infty$ cannot happen. We may also assume that both $o_{1/\rho}$ and o_ψ are finite. For any positive δ, we now have

$$\rho(x) = \Omega(x^{o_{1/\rho}-\delta}) \qquad \text{and} \qquad 1/\psi(x) = \Omega(x^{o_\psi-\delta}).$$

We choose $\alpha(x) = \Theta(x^{-(1/p+\delta)})$ and conclude

$$\int_R^x \frac{(x-t)^{r-1}}{(r-1)!} \frac{|\alpha(t)|}{\psi(t)}\, dy = \Omega\left(x^{r-1} \int_R^{x/2} t^{o_\psi-1/p-2\delta}\, dt \right)$$
$$= \Omega(x^{r-1+\max\{0,\,o_\psi-2\delta+1-1/p\}}).$$

From this we obtain

$$L^q(R) = \Omega\left(\int_R^\infty x^{q(o_{1/\rho}-\delta+r-1+\max\{0,\,o_\psi-2\delta+1-1/p\})}\, dx \right).$$

For sufficiently small δ, the exponent $q(o_{1/\rho}-\delta+r-1+\max\{0,\, o_\psi-2\delta+1-1/p\})>-1$. Indeed, the last inequality is equivalent to $-o_{1/\rho}+\min\{0,\, -o_\psi-1+1/p\}<r-1+1/q$ which holds due to the hypotheses of the theorem. Therefore, $L(R)=\infty$ for every $R\geqslant 0$. This implies that the complexity is infinite for small ε. This completes the proof. ▮

Theorem 2 simplifies if we assume that $o_\rho=-o_{1/\rho}$ and $o_\psi=-o_{1/\psi}$ and both are finite numbers. Then we can rewrite Theorem 2 as

$$o_\rho+\min\{1-1/p,\, o_{1/\psi}\}>\gamma \quad \text{implies} \quad \mathrm{comp}(\varepsilon)<\infty,\ \forall\varepsilon>0, \tag{19}$$

$$o_\rho+\min\{1-1/p,\, o_{1/\psi}\}<\gamma \quad \text{implies} \quad \mathrm{comp}(\varepsilon)=\infty, \text{ for small } \varepsilon>0. \tag{20}$$

In the next section we show that (19) holds with $\mathrm{comp}(\varepsilon)$ of order $\varepsilon^{-1/s}$, which is the complexity of the classical approximation problem.

In view of (19) and (20), it is interesting to ask what happens for

$$o_\rho+\min\{1-1/p,\, o_{1/\psi}\}=\gamma \tag{21}$$

It is easy to check that in this case, the complexity may be finite or infinite depending on the specific form of the weights ρ and ψ. For example, take $\psi\equiv 1$. Then $o_\psi=o_{1/\psi}=0$. For $\rho(t)=(2+|t|)^{-\gamma}\ln^{-2\gamma}(2+|t|)$ we have $o_\rho=\gamma$ and (21) holds. From (16) we conclude that

$$\tilde{L}(R)=O\left(\int_R^\infty (x\ln^2 x)^{-1}\,dx\right)=O\left(\int_{\ln R}^\infty t^{-2}\,dt\right)=O(\ln^{-1}R).$$

This implies finite complexity.

On the other hand for the same ψ, take $\rho(t)=(2+|t|)^{-\gamma}\ln^{2/p}(2+|t|)$. Then once more $o_\rho=\gamma$ and (21) holds. For $\alpha(t)=\Theta((2+|t|)^{-1/p}\ln^{-2/p}(2+|t|))$ we have

$$L(R)=\Omega\left(\int_R^\infty x^{-1}\,dx\right)=\infty.$$

Hence, the complexity is infinite for small ε.

We also illustrate Theorem 2 for the weight functions $\rho=\psi$ with $o_\rho=-o_{1/\rho}$. Assume first that o_ρ is a finite number. Then (ii) of Theorem 2 is satisfied, since $o_\rho+\min\{1-1/p,\, -o_\rho\}\leqslant 0<\gamma$. Hence, for such weights, the complexity of weighted approximation is infinite for small ε. For $o_\rho=\infty$, complexity can be either finite or infinite, depending on the form of the weight ρ. Indeed, take $p=q=\infty$. Then Case 1 yields that for $\rho(t)=\exp(-|t|)$ we have infinite complexity, and for $\rho(t)=\exp(-t^2)$ we have finite complexity.

4. EQUIVALENCE OF COMPLEXITIES

In this section we address the question as to when the complexity of weighted approximation is of the same order as the complexity of classical approximation, that is, when it is of order $\varepsilon^{-1/s}$ with $s=\gamma$ for $p \leqslant q$ and $s=r$ for $p>q$; see (4).

We first present an algorithm for solving the weighted approximation problem for monotonic weights that computes an ε-approximation with cost proportional to $\varepsilon^{-1/s}$. Then we show how this algorithm can be used for general weights satisfying (i) of Theorem 2.

4.1. *An Algorithm for Monotonic Weights*

In this subsection we assume that both ρ and ψ are monotonic on $\mathbb{R}_+$. That is, we assume that ρ is nonincreasing since Corollary 1 implies that we would have infinite complexity otherwise. The weight ψ can be either nonincreasing or nondecreasing.

Define $\kappa\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\kappa(x) = \frac{\rho(x)}{\psi(x)} \quad \text{if } \psi \text{ is nondecreasing,} \qquad \text{and} \qquad \kappa(x) = \frac{\rho(x)}{\psi(2x)} \quad \text{otherwise.}$$

We also assume that κ is nonincreasing, that $\kappa^{1/\gamma}$ is integrable, i.e.,

$$\int_1^\infty \kappa^{1/\gamma}(x)\, dx < \infty, \tag{22}$$

and that there exist positive constants A_2 and A_3 such that

$$L(R) \leqslant A_2 \kappa(R)\, R^\gamma, \qquad \forall R \geqslant 2, \tag{23}$$

and

$$i2^i \kappa^{1/\gamma}(2^i) \leqslant A_3/\ln 2, \qquad \forall i \geqslant 1. \tag{24}$$

Later, we shall need the quantity

$$A := \kappa^{1/\gamma}(0) + \int_1^\infty \kappa^{1/\gamma}(x)\, dx.$$

Some of the above assumptions could be relaxed as discussed in the second half of this section. We decided to start with stronger assumptions since they are satisfied by a number of important families of weights. They also allow us to get explicit estimates of the error and cost of the proposed algorithm, avoiding the Θ-notation.

We are ready to define the algorithm $\mathscr{U}=\mathscr{U}_\varepsilon$. Let $a_0=0$, $a_i=2^i$ and $a_{-i}=-a_i$ for $i\geqslant 1$. For a given positive integer k, let $I_i=[a_{i-1}, a_i]$ and $I_{-i}=-I_i$ for $i=1, ..., k$. For $|i|=k+1$ we have $I_{k+1}=[2^k, \infty)$ and $I_{-(k+1)}=-I_{k+1}=(-\infty, -2^k]$. We denote the lengths of the intervals $I_{\pm i}$ by r_i. Hence $r_1=2$ and $r_i=a_{i-1}=2^{i-1}$ for $i=2, ..., k$.

The algorithm $\mathscr{U}$ depends on integer parameters k and $m_1, ..., m_k\geqslant 1$; their choice will be discussed below. On the interval $I_{\pm i}$, $i=1, ..., k$, the algorithm $\mathscr{U}$ uses m_i+r values at equally spaced points and is an interpolating piecewise polynomial of degree $r-1$, see Remark 1. For $x\in I_{k+1}$ or $x\in I_{-k-1}=-I_{k+1}$, the algorithm $\mathscr{U}$ is the Taylor polynomial of degree $r-1$ with the center at a_k or $-a_k$, respectively. Note that the total cardinality of the information used by $\mathscr{U}$ equals

$$\mathrm{card}(\mathscr{U})=2r(k+1)-2k+1+2\sum_{i=1}^{k} m_i. \tag{25}$$

Define

$$c_1=\kappa(0)\,2^\gamma \qquad \text{and} \qquad c_i=\kappa(2^{i-1})\,2^{\gamma(i-1)} \qquad \text{for} \quad i=2, 3, ...$$

Note that $\int_1^\infty \kappa^{1/\gamma}(x)\,dx=\sum_{i=0}^\infty \int_{2^i}^{2^{i+1}} \kappa^{1/\gamma}(x)\,dx$. Since $\kappa^{1/\gamma}$ is monotone, we have

$$\frac{1}{2}\sum_{i=0}^{\infty} \kappa^{1/\gamma}(2^{i+1})\,2^{i+1}\leqslant\int_1^\infty \kappa^{1/\gamma}(x)\,dx\leqslant\sum_{i=0}^{\infty} \kappa^{1/\gamma}(2^i)\,2^i.$$

Note that (22) implies $\sum_{i=1}^\infty c_i^{1/\gamma}<\infty$, and that (24) implies

$$c_{i+1}\leqslant\frac{A_3^\gamma}{i^\gamma \ln^\gamma 2}.$$

The choice of the parameters k and $m_1, ..., m_k$ depends on the relation between p and q.

Case $p\leqslant q$. Given an error demand ε, where $0<\varepsilon<1$, we take $k=k(\varepsilon)$ as the minimal integer for which

$$c_{k+1}\leqslant\varepsilon\max\{A_1, A_2\},$$

where A_1 is given in Remark 1. This means that $\varepsilon/\max\{A_1, A_2\}<c_k$ and, due to the properties of c_i, such k is well-defined. Moreover,

$$k(\varepsilon)\leqslant 1+\frac{A_3(\max\{A_1, A_2\})^{1/\gamma}}{\varepsilon^{1/\gamma}\ln 2}.$$

The numbers m_i are chosen as

$$m_i = m_i(\varepsilon) = \left\lceil \left(\max\{A_1, A_2\} \, \frac{c_i}{\varepsilon} \right)^{1/\gamma} \right\rceil \qquad \text{for} \quad i = 1, ..., k+1.$$

Observe that $m_{k+1} = 1$.

Case $p > q$. We now select k and the m_i's as follows. Define

$$C(\varepsilon, \ell) = \left(\frac{\max\{A_1, A_2\}}{\varepsilon} \right)^{1/r} \left(2 \sum_{i=1}^{\ell+1} c_i^{1/\gamma} \right)^{(1-q/p)/(qr)}$$

for $\varepsilon \in (0, 1]$ and $\ell \geqslant 1$. Observe that $C(\varepsilon, \ell) \leqslant C(\varepsilon, \infty) < \infty$. We take $k = k(\varepsilon)$ as the minimal integer for which

$$c_{k+1}^{1/\gamma} \, C(\varepsilon, k) \leqslant 1.$$

Note that $k(\varepsilon)$ is well-defined and

$$k(\varepsilon) \leqslant 1 + \frac{A_3(\max\{A_1, A_2\})^{1/r}}{\varepsilon^{1/r} \ln 2} \left(2 \sum_{i=1}^{\infty} c_i^{1/\gamma} \right)^{(1-q/p)/(qr)}.$$

The numbers m_i are chosen as

$$m_i = m_i(\varepsilon) = \lceil c_i^{1/\gamma} \, C(\varepsilon, k) \rceil \qquad \text{for} \quad i = 1, ..., k+1.$$

As before, $m_{k+1} = 1$.

This concludes the definition of the algorithm $\mathcal{U} = \mathcal{U}_\varepsilon$. We now obtain the estimates of the error and cardinality of information used by the algorithm $\mathcal{U}_\varepsilon$.

THEOREM 3. *For every* $\varepsilon \in (0, 1]$ *we have* $e(\mathcal{U}_\varepsilon, \rho, \psi, q, p) \leqslant \varepsilon$ *and*

$$\operatorname{card}(\mathcal{U}_\varepsilon) \leqslant 4r + 1 + (\max\{A_1, A_2\})^{1/s} \times \left((4A)^{\gamma/s} + \frac{2r}{\ln 2} A_3 (4A)^{(1/q - 1/p)_+/r} \right) \varepsilon^{-1/s}.$$

Proof. Denote by $e_i(f)$ the errors of $\mathcal{U} = \mathcal{U}_\varepsilon$ for a function f restricted to I_i. That is,

$$e_i(f) = \left(\int_{I_i} |f(x) - \mathcal{U}(f)(x)|^q \, \rho^q(x) \, dx \right)^{1/q}.$$

Then

$$\left(\int_{\mathbb{R}} |f(x) - \mathscr{U}(f)(x)|^q \, \rho^q(x)\, dx\right)^{1/q} = \left(\sum_{1 \leqslant |i| \leqslant k+1} e_i^q(f)\right)^{1/q}.$$

For $|i| \in [1, k]$, the monotonicity of ρ and Remark 1 imply that

$$e_{\pm i}(f) \leqslant A_1 \rho(a_{i-1}) \frac{r_i^{\gamma}}{m_i^s} \, b_{\pm i}(f),$$

where (for all $|i| \leqslant k+1$)

$$b_i(f) = \left(\int_{I_i} |f^{(r)}(x)|^p \, dx\right)^{1/p} = \left(\int_{I_i} |f^{(r)}(x)\, \psi(x)|^p \, \psi^{-p}(x)\, dx\right)^{1/p}$$

$$\leqslant \frac{d_i(f)}{\psi(a'_{i-1})} \qquad \text{with} \quad d_i(f) = \left(\int_{I_i} |f^{(r)}(x)\, \psi(x)|^p \, dx\right)^{1/p}.$$

Here $a'_{i-1} = a_{i-1}$ if ψ is nondecreasing, and $a'_{i-1} = a_i$ otherwise. Hence,

$$e_{\pm i}(f) \leqslant A_1 c_i d_{\pm i}(f)\, m_i^{-s}.$$

Of course, for $i = k+1$, we have as in (14),

$$e_{\pm(k+1)}(f) \leqslant L(a_k)\, d_{\pm(k+1)}(f) \leqslant A_2 c_{k+1} d_{\pm(k+1)}(f)$$

due to (23). Since $m_{k+1} = 1$, we have

$$e_{\pm i}(f) \leqslant \max\{A_1, A_2\}\; c_i d_{\pm i}(f)\, m_i^{-s} \qquad \text{for} \quad i = 1, 2, ..., k+1. \tag{26}$$

Using the fact that $(\sum_{i=1}^{k+1} d^p_{\pm i}(f))^{1/p}$ is bounded by 1, we obtain

$$e(\mathscr{U}) = \sup_{f \in \mathscr{F}_p} \left(\sum_{1 \leqslant |i| \leqslant k+1} e_i^q(f)\right)^{1/q}$$

$$\leqslant \max\{A_1, A_2\} \sup \left\{ \left(\sum_{1 \leqslant |i| \leqslant k+1} d_i^q c_{|i|}^q m_{|i|}^{-qs}\right)^{1/q} : \right.$$

$$\left. d_i \geqslant 0, \left(\sum_{1 \leqslant |i| \leqslant k+1} d_i^p\right)^{1/p} \leqslant 1 \right\}.$$

Let $w_i = c_{|i|} m_{|i|}^{-s}$. Then the last supremum is the operator norm of a diagonal operator with weights w_i. For completeness we estimate this norm.

Consider the case $p \leqslant q$. Since $d_i \leqslant 1$ then $d_i^q \leqslant d_i^p$ and

$$\left(\sum_{1 \leqslant |i| \leqslant k+1} d_i^q w_i^q \right)^{1/q} \leqslant \max_{1 \leqslant i \leqslant k+1} w_i \left(\sum_{1 \leqslant |i| \leqslant k+1} d_i^p \right)^{1/q} \leqslant \max_{1 \leqslant i \leqslant k+1} w_i.$$

For $p > q$, we use the Hölder inequality with $1/p' + q/p = 1$, i.e., $p' = 1/(1-q/p)$, and obtain

$$\sum_{1 \leqslant |i| \leqslant k+1} d_i^q w_i^q \leqslant \left(2 \sum_{i=1}^{k+1} w_i^{q/(1-q/p)} \right)^{1-q/p} \left(\sum_{1 \leqslant |i| \leqslant k+1} d_i^p \right)^{q/p}$$
$$\leqslant \left(2 \sum_{i=1}^{k+1} w_i^{q/(1-q/p)} \right)^{1-q/p}.$$

This shows that

$$e(\mathscr{U}) \leqslant \max\{A_1, A_2\} \times \begin{cases} \max_{1 \leqslant i \leqslant k+1} c_i m_i^{-s} & \text{if} \quad p \leqslant q, \\ \left(2 \sum_{i=1}^{k+1} (c_i m_i^{-s})^{q/(1-q/p)} \right)^{1/q - 1/p} & \text{if} \quad p > q. \end{cases}$$

Consider now the case $p \leqslant q$. Then $s = \gamma = r + 1/q - 1/p$. From the definition of m_i we have

$$c_i m_i^{-\gamma} \leqslant \varepsilon_1 := \varepsilon / \max\{A_1, A_2\}.$$

This yields $e(\mathscr{U}) \leqslant \varepsilon$, as claimed.

We use (25) to estimate the cardinality $\operatorname{card}(\mathscr{U})$. Note that $\sum_{i=1}^k m_i \leqslant k + \varepsilon_1^{-1/\gamma} \sum_{i=1}^k c_i^{1/\gamma}$. We now show that

$$\sum_{i=1}^k c_i^{1/\gamma} \leqslant 2A.$$

Indeed, the monotonicity of $\kappa^{1/\gamma}$ implies that

$$\sum_{i=1}^k c_i^{1/\gamma} = 2\kappa^{1/\gamma}(0) + \sum_{i=2}^k 2^{i-1} \kappa^{1/\gamma}(2^{i-1})$$
$$\leqslant 2\kappa^{1/\gamma}(0) + 2 \sum_{i=2}^k \int_{2^{i-2}}^{2^i - 1} \kappa^{1/\gamma}(t)\, dt$$
$$\leqslant 2\kappa^{1/\gamma}(0) + 2 \int_1^\infty \kappa^{1/\gamma}(t)\, dt = 2A,$$

as claimed. From this we conclude that $\sum_{i=1}^{k} m_i \leqslant k + 2A\varepsilon_1^{-1/\gamma}$, and

$$\mathrm{card}(\mathscr{U}) \leqslant 2r(k+1) + 1 + 4A\varepsilon_1^{-1/\gamma}.$$

Since $k(\varepsilon) \leqslant 1 + A_3\varepsilon^{-1/\gamma}/\ln 2$, this completes the proof for $p \leqslant q$.

We now consider the case $p > q$. Then $s = r$ and $m_i \geqslant c_i^{1/\gamma} C(\varepsilon, k)$, which implies that $c_i m_i^{-s} \leqslant c_i^{1-r/\gamma} C^{-r}(\varepsilon, k)$. Note also that $1 - r/\gamma = (1-q/p)/(q\gamma)$. Therefore

$$\begin{aligned}\frac{e(\mathscr{U})}{\max\{A_1, a_2\}} &\leqslant \left(2 \sum_{i=1}^{k+1} (c_i m_i^{-s})^{q/(1-q/p)}\right)^{(1-q/p)/q} \\ &\leqslant \left(2 \sum_{i=1}^{k+1} (c_i^{1-r/\gamma} C^{-r}(\varepsilon, k))^{q/(1-q/p)}\right)^{(1-q/p)/q} \\ &= C^{-r}(\varepsilon, k)\left(2 \sum_{i=1}^{k+1} c_i^{1/\gamma}\right)^{1/q-1/p} = \varepsilon_1.\end{aligned}$$

Hence $e(\mathscr{U}) \leqslant \varepsilon$, as claimed. The estimate of $\mathrm{card}(\mathscr{U})$ is similar to that for the case $p \leqslant q$, and we omit this part. ∎

Theorem 3 states that the cost of the algorithm $\mathscr{U}_\varepsilon$ is of the same order as the complexity of the classical approximation problem. This means that the algorithm $\mathscr{U}_\varepsilon$ is optimal (up to a multiplicative factor) and the complexity of weighted approximation is of the same order as the complexity of classical approximation.

This holds for monotonic weights satisfying (22)–(24). In the rest of this subsection we discuss whether the conditions (22)–(24) are necessary. We begin with the following two remarks.

Remark 3. The monotonicity assumption can be relaxed in a number of ways. One can relax it by assuming the existence of weights $\hat{\rho}$ and $\hat{\psi}$ satisfying the assumptions (22)–(24) and such that $\rho(x) \leqslant \hat{\rho}(x)$ and $\psi(x) \geqslant \hat{\psi}(x)$ for all x. Then the algorithm derived for such new weights will still yield approximations with the error bounded by ε and cost proportional to $\varepsilon^{-1/s}$. One could also handle even more general weights by computing the maxima of ρ and the minima of ψ for each subinterval $[a_{i-1}, a_i]$ and defining m_i as the ratio of κ at these maximum and minimum points. This, however, can be prohibitively expensive.

Finally, an easy relaxation is to assume monotonicity of the weights only for sufficiently large arguments. Similarly, the complexity of the weighted problem remains proportional to $\varepsilon^{-1/s}$ if (23) holds for sufficiently large R (not necessarily for $R \geqslant 2$).

Remark 4. Recall that we use the argument $2x$ in the definition of κ when ψ is nonincreasing. Moreover, the inequality (24) is expressed in terms of powers of two. We have chosen the constant 2 only for the sake of simplicity. Indeed, it can be replaced by any number $z>1$. That is, we could define $\kappa(x)=\rho(x)/\psi(zx)$ (when ψ is nonincreasing) and restate (24) as

$$\kappa^{1/\gamma}(z^i)\, z^i \ln z^i \leqslant A_3, \qquad \forall i \geqslant 1.$$

Of course, then we should use $a_i = z^i$ instead of 2^i in the definition of the algorithm U_ε. The importance of this remark may be illustrated by choosing weight functions $\rho(x)=e^{-ax}$ and $\psi(x)=e^{-bx}$, where (necessarily) $a>0$. We can conclude then that the complexity of the weighted problem is proportional to $\varepsilon^{-1/s}$ if $a>b$, since the modified assumption (22)–(24) hold with any $z \in (1, a/\max\{0,b\})$. As we have already mentioned, we have infinite complexity when $a \leqslant b$, see also the following proposition.

We now discuss the necessity of assumption (22).

PROPOSITION 1. *Suppose that ρ is monotonically nonincreasing and ψ is monotonic. Then the condition* (22) *is necessary for* $\mathrm{comp}(\varepsilon, \rho, \psi) = \Theta(\varepsilon^{-1/s})$.

Proof. Let $\mathrm{comp}(\varepsilon) = \Theta(\varepsilon^{-1/s})$. There exists an algorithm $\mathcal{U} = \mathcal{U}_\varepsilon$ that uses $n = n(\varepsilon) = \mathrm{comp}(\varepsilon) \leqslant c_1 \varepsilon^{-1/s}$ sample points and has error at most ε. In what follows, let c_i denote various positive numbers that are independent of ε. Let $x_i = x_i(\varepsilon)$ be the positive sample points used by $\mathcal{U}$, where $0 = x_0 < x_1 < \cdots < x_k$ with $k = k(\varepsilon) \leqslant n$.

Let $h_i(t) = h_i(t;\varepsilon) = (t - x_{i-1})_+^r\,(x_i - t)_+^r$ and $\hat{h}_i(t) = h_i(t)/\|h_i^{(r)}\psi\|_p$. Obviously, $\pm \hat{h}_i \in \mathcal{F}_p$. The information about $\pm \hat{h}_i$ is zero, and therefore the error of $\mathcal{U}$ cannot be smaller than the weighted $L_q(\mathbb{R})$ norm of $\hat{h}_i$; see also (2) with $R = \infty$ and with the weighted norms. Hence,

$$\varepsilon \geqslant \left(\int_{\mathbb{R}} \hat{h}_i^q(t)\, \rho(t)\, dt \right)^{1/q} = \left(\frac{\int_{x_{i-1}}^{x_i} h_i^q(t)\, \rho^q(t)\, dt}{\left(\int_{x_{i-1}}^{x_i} |h_i^{(r)}(t)\, \psi(t)|^p\, dt \right)^{q/p}} \right)^{1/q}. \tag{27}$$

It is easy to verify that

$$\left(\frac{\int_{x_{i-1}}^{x_i} h_i^q(t)\, dt}{\left(\int_{x_{i-1}}^{x_i} |h_i^{(r)}(t)|^p\, dt \right)^{q/p}} \right)^{1/q} = c_2 (x_i - x_{i-1})^\gamma.$$

Define

$$\tilde{\kappa}_i = \tilde{\kappa}_i(\varepsilon) = \frac{\min_{x_{i-1} \leqslant t \leqslant x_i} \rho(t)}{\max_{x_{i-1} \leqslant t \leqslant x_i} \psi(t)},$$

which equals $\rho(x_i)/\psi(x_i)$ if ψ is nondecreasing and $\rho(x_i)/\psi(x_{i-1})$ otherwise. From the monotonicity of the weights, we get that

$$\varepsilon \geqslant \tilde{\kappa}_i \left(\frac{\int_{x_{i-1}}^{x_i} h_i^q(t)\, dt}{\left(\int_{x_{i-1}}^{x_i} |h_i^{(r)}(t)|^p\, dt \right)^{q/p}} \right)^{1/q} = c_2 \tilde{\kappa}_i (x_i - x_{i-1})^{\gamma}, \tag{28}$$

or equivalently that

$$\tilde{\kappa}_i^{1/\gamma}(x_i - x_{i-1}) \leqslant c_3 \varepsilon^{1/\gamma}.$$

This, together with (9) and (10), implies that for any positive number M, we have

$$\lim_{\varepsilon \to 0} \max\{x_i(\varepsilon) - x_{i-1}(\varepsilon) : x_{i-1}(\varepsilon) \leqslant M\} = 0.$$

Define

$$I(\varepsilon) := \sum_{i=1}^{k(\varepsilon)} \tilde{\kappa}_i^{1/\gamma}(x_i - x_{i-1}),$$

and

$$k(\varepsilon, M) := \min\{i \colon x_i(\varepsilon) \geqslant M\} \quad \text{and} \quad I(\varepsilon, M) := \sum_{i=1}^{k(\varepsilon, M)} \tilde{\kappa}_i^{1/\gamma}(x_i - x_{i-1}).$$

Note that $k(\varepsilon, M)$ exists and converges to infinity with ε going to zero.

We need to prove that $I(\varepsilon)$ is uniformly bounded, i.e., that there exists a positive constant c_4 such that

$$I(\varepsilon) \leqslant c_4, \qquad \forall \varepsilon > 0. \tag{29}$$

To this end, consider first the case $p \leqslant q$. Since there are $k(\varepsilon) \leqslant c_1 \varepsilon^{-1/s}$ such points and $s = \gamma$, we have

$$I(\varepsilon) = \sum_{i=1}^{k(\varepsilon)} \tilde{\kappa}_i^{1/\gamma}(x_i(\varepsilon) - x_{i-1}(\varepsilon)) \leqslant c_3 \varepsilon^{1/\gamma} n(\varepsilon) \leqslant c_4, \qquad \forall \varepsilon > 0.$$

Consider now the case $p>q$. Then $s=r<\gamma$ and $k=k(\varepsilon)\leqslant c_1\varepsilon^{-1/r}$. We first show that

$$a=a(\varepsilon):=\left(\sum_{i=1}^{k}\left(\tilde{\kappa}_i(x_i-x_{i-1})^{\gamma}\right)^{q/(1-q/p)}\right)^{(1-q/p)/q}\leqslant c_5\varepsilon. \tag{30}$$

Indeed, consider the function

$$f(x)=\rho(x)\sum_{i=1}^{k}d_i\hat{h}_i(x)$$

for d_i chosen such that $(\sum_{i=1}^{k}|d_i\psi(a_i')|^p)^{1/p}\leqslant 1$ with $a_i'=x_i$ for nondecreasing ψ and $a_i'=x_{i-1}$ otherwise. Clearly, $f\in\mathscr{F}_p$ and has zero values at the sample points x_i. Therefore its weighted $L_q(\mathbb{R})$ norm must be at most ε. Since the weights are monotonic, we have

$$\begin{aligned}\varepsilon&\geqslant\left(\sum_{i=1}^{k}|d_i|^q\int_{x_{i-1}}^{x_i}\hat{h}_i^q(t)\,\rho^q(t)\,dt\right)^{1/q}\\&\geqslant c_6\left(\sum_{i=1}^{k}|d_i\psi(a_i')|^q\,(\tilde{\kappa}_i(x_i-x_{i-1})^{\gamma})^q\right)^{1/q}.\end{aligned}$$

We maximize the right hand side with respect to c_i by taking c_i proportional to $(\kappa(x_i)(x_i-x_{i-1})^{\gamma})^{1/(1-q/p)}/\psi(a_i')$ and obtain (30). We now apply Hölder's inequality to $I(\varepsilon)$ to get

$$I(\varepsilon)\leqslant a^{1/\gamma}k(\varepsilon)^{1/u}\qquad\text{with}\quad\frac{1}{u}=1-\frac{1-q/p}{\gamma q}=\frac{r}{\gamma}.$$

Hence, as claimed in (29), $I(\varepsilon)\leqslant c_7\varepsilon^{1/\gamma}\varepsilon^{-1/\gamma}\leqslant c_4$.

Since $I(\varepsilon, M)\leqslant I(\varepsilon)$, we conclude from (29) that $I(\varepsilon, M)$ is uniformly bounded:

$$I(\varepsilon, M)\leqslant c_4,\qquad\forall\varepsilon>0,\quad\forall M>0.$$

Since for nondecreasing ψ we have

$$\lim_{\varepsilon\to 0}I(\varepsilon, M)=\int_0^M\left(\frac{\rho(x)}{\psi(x)}\right)^{1/\gamma}dx,$$

this implies that (22) holds. The equality above also holds for decreasing ψ. Indeed, for monotonic weights, we can assume their continuity. Then these weights are uniformly continuous over $[0, M]$ and $|\tilde{\kappa}_i^{1/\gamma}(\varepsilon)-(\rho(x_i(\varepsilon))/\psi(x_i(\varepsilon)))^{1/\gamma}|$ converges to zero uniformly in i. Since $\sum_{i=1}^{k(\varepsilon, M)}(x_i(\varepsilon)-x_{i-1}(\varepsilon))$ converges to M, this completes the proof. ∎

Remark 5. From the proof of Proposition 1 it is clear that the monotonicity assumption on ρ and ψ can be relaxed by assuming only continuity of the weight functions.

We end this subsection by the following proposition that addresses the assumption (23).

PROPOSITION 2. *Suppose that* $L(1)$ *is finite and there exist positive constants* c_1 *and* c_2 *such that*

$$\rho(xR) \leqslant c_1 \rho(x)\, \rho(R) \qquad \text{and} \qquad \psi(x)\, \psi(R) \leqslant c_2 \psi(xR) \tag{31}$$

for every $x \geqslant 1$ *and every sufficiently large* R. *Then* (23) *holds with* $A_2 = c_1 L(1)/c_2 < \infty$.

Proof. The inequality (23) follows directly from (12). ▮

Note that (31) holds for $\rho(x) = (1+x)^{-a}$ $(a>0)$ with $c_1 = 2^a$ and $\psi(x) = (1+x)^{-b}$ with $c_2 = 2^{\min\{0,\, b\}}$.

4.2. *Equivalence and Orders*

In this subsection we relate the equivalence of the complexity to the orders of the weight functions at infinity. We begin with the following theorem.

THEOREM 4. *Assume that the set* $\{o_\rho, o_{1/\psi}\}$ *is different from the set* $\{\infty, -\infty\}$. *Then the condition* (17) *of Theorem* 2,

$$o_\rho + \min\{1 - 1/p,\, o_{1/\psi}\} > \gamma$$

implies that

$$\mathrm{comp}(\varepsilon, \rho, \psi) = \Theta(\varepsilon^{-1/s}) \qquad as \quad \varepsilon \to 0.$$

Proof. The proof is based on replacing $\rho(x)$ by its upper bound $\tilde{\rho}(x) = c_1(1+x)^{-o_\rho + \delta}$ and $\psi(x)$ by its lower bound $\tilde{\psi}(x) = c_2(1+x)^{o_{1/\psi} - \delta}$, where δ is a positive number chosen to guarantee that $o_\rho - \delta + \min\{1 - 1/p, o_{1/\psi} - \delta\} > \gamma$. The weights $\tilde{\rho}$ and $\tilde{\psi}$ satisfy the assumptions (22)–(24) for sufficiently large x which is enough due to Remark 3. ▮

We now discuss the necessity of the assumption, $\alpha := o_\rho + \min\{1 - 1/p, o_{1/\psi}\} > \gamma$, in Theorem 4. For simplicity of presentation we now assume that $o_\rho = -o_{1/p}$ and $o_\psi = -o_{1/\psi}$, and both are finite. From Theorem 2, we know that the complexity is infinite if $\alpha < \gamma$. It is therefore natural to ask what happens when $\alpha = \gamma$. As stated in the following proposition anything can happen even for a relatively simple special case.

PROPOSITION 3. *Let* $r=1$, $p=q=\infty$, $\psi(x)=1$, *and* $\rho(x)=(x+e)^{-1}\ln^{-a}(x+e)$. *Then* $o_\rho+\min\{1-1/p, o_{1,\psi}\}=\gamma=1$ *and*

$$\operatorname{comp}(\varepsilon,\rho,\psi)=\begin{cases}\Theta(\varepsilon^{-1}) & \textit{if} \quad a>1,\\ \Theta(\varepsilon^{-1}\ln\varepsilon^{-1}) & \textit{if} \quad a=1,\\ \Theta(\varepsilon^{-1/a}) & \textit{if} \quad 0<a<1,\\ \infty & \textit{if} \quad a\leqslant 0.\end{cases}$$

Proof. For $a>1$, this follows from the fact that (22)–(24) hold. For $a\leqslant 0$, this follows from Theorem 1 since Case 1 of Section 3 implies that

$$L(R)=\sup_{x\geqslant R}\frac{x-R}{(x+e)\ln^a(x+e)}\sim\ln^{|a|}R$$

does not converge to zero.

We now prove the proposition for $a\in(0,1]$. Given $\varepsilon\in(0,1)$, let $x_1, ..., x_n$ be the non-negative points of information used by an optimal algorithm whose error does not exceed ε. Without loss of generality we can assume that $x_1=0$ and that for negative arguments, the algorithm uses $-x_2, ..., -x_n$. Of course, the number n and the points x_i depend on ε.

Since $1/\rho$ is convex, it is easy to check, see also Appendix G in [7], that the optimal location of the information points is such that

$$\frac{x_{i+1}-x_i}{2}\,\rho\left(\frac{x_{i+1}+x_i}{2}\right)=\varepsilon \tag{32}$$

and that $n=n(\varepsilon)$ is the first index for which $L(x_n)\leqslant\varepsilon$. From the form of ρ, we have that

$$x_{n(\varepsilon)}=\exp(\varepsilon^{-1/a}(1+o(1))).$$

Let $m=m(\varepsilon)$ be the largest index for which

$$x_m+e\leqslant\exp((1/(6\varepsilon))^{1/a}).$$

Since $x_{n(\varepsilon)}+e\geqslant 0.5\exp((2\varepsilon)^{-1/a})$ for small $\varepsilon>0$, we have that

$$m\leqslant n(\varepsilon) \qquad \text{and} \qquad x_{m(\varepsilon)}\geqslant\exp((1/(7\varepsilon))^{1/a}) \quad \text{for sufficiently small } \varepsilon.$$

For $i\leqslant m-1$, we conclude from (32) that

$$\varepsilon\geqslant\frac{x_{i+1}-x_i}{2}\,\rho(x_{i+1})\geqslant\frac{x_{i+1}-x_i}{2(e+x_{i+1})}\,6\varepsilon.$$

Hence, $x_{i+1}-x_i \leqslant (e+x_{i+1})/3$ and $x_{i+1} \leqslant 1.5x_i + e/2$. This yields

$$\frac{x_{i+1}-x_i}{e+x_i} \leqslant \frac{1}{2}. \tag{33}$$

For small ε we have

$$\begin{aligned}(n(\varepsilon)-1)\,\varepsilon \geqslant (m(\varepsilon)-1)\,\varepsilon &= \sum_{i=1}^{m(\varepsilon)-1} \frac{x_{i+1}-x_i}{2}\,\rho\left(\frac{x_{i+1}+x_i}{2}\right)\\ &= \sum_{i=1}^{m(\varepsilon)-1} \frac{x_{i+1}-x_i}{2}\,\rho(x_i)\\ &\quad + \sum_{i=1}^{m(\varepsilon)-1} \frac{x_{i+1}-x_i}{2}\left(\rho\left(\frac{x_{i+1}+x_i}{2}\right)-\rho(x_i)\right).\end{aligned}$$

Note that $\rho'(x) = -\rho(x)/(x+e)(1+a/\ln(x+e)) \geqslant -2\rho(x)/(x+e)$. This and (33) yield

$$\rho\left(\frac{x_{i+1}+x_i}{2}\right)-\rho(x_i) \geqslant -\frac{2\rho(x_i)}{e+x_i}\,\frac{x_{i+1}-x_i}{2} \geqslant -\frac{\rho(x_i)}{2}.$$

Hence,

$$(n(\varepsilon)-1)\,\varepsilon \geqslant \frac{1}{2}\sum_{i=1}^{m(\varepsilon)-1} \frac{x_{i+1}-x_i}{2}\,\rho(x_i) \geqslant \frac{1}{4}\int_{x_1}^{x_{m(\varepsilon)}} \rho(t)\,dt.$$

As already mentioned, $x_m \geqslant \exp((1/(7\varepsilon))^{1/a})$ with ε approaching zero. Therefore

$$n(\varepsilon) \geqslant 1 + \frac{1}{4\varepsilon}\int_0^{\exp((1/(7\varepsilon))^{1/a})} \rho(t)\,dt.$$

Since the integral is proportional to $\varepsilon^{-(1-a)/a}$ for $a<1$ and $\ln\varepsilon^{-1}$ for $a=1$, this completes the proof of the lower bound on the complexity.

To show an upper bound on $(n(\varepsilon)-1)\,\varepsilon$ observe that

$$\begin{aligned}(n(\varepsilon)-1)\,\varepsilon &= \sum_{i=2}^{n(\varepsilon)} \frac{x_{i+1}-x_i}{2}\,\rho\left(\frac{x_{i+1}+x_i}{2}\right)\\ &\leqslant \sum_{i=2}^{n(\varepsilon)} \int_{x_i}^{(x_{i+1}+x_i)/2} \rho(t)\,dt \leqslant \int_0^{x_{n(\varepsilon)}} \rho(t)\,dt.\end{aligned}$$

Since the last integral is of the same order as the integral for the lower bound, the proof is complete. ∎

Proposition 3 presents an example of the weighted approximation problem with complexity that is polynomial in ε^{-1} for $a > 0$. Using the same proof technique it is easy to show that the complexity of weighted approximation can also be an exponential function of ε^{-1}. This is presented in the following proposition.

PROPOSITION 4. *Let* $r = 1$, $p = q = \infty$, $\psi(x) = 1$, *and* $\rho(x) = (x+e)^{-1} \ln^{-1} \ln(x + e^e)$. *Then* $o_\rho + \min\{1 - 1/p, o_{1/\psi}\} = \gamma = 1$ *and*

$$\mathrm{comp}(\varepsilon, \rho, \psi) = \Theta(\exp(\varepsilon^{-1}(1 + o(1)))).$$

We think that in the general case with $\alpha = \gamma$ the complexity depends on the integral of $\kappa^{1/\gamma}$ over the interval of length proportional to $L^{-1}(\varepsilon)$. This is the case when the weights satisfy the assumptions (22)–(24), as well as for problems in Propositions 3 and 4. We propose therefore the following conjecture.

Conjecture 1. If

$$o_\rho + \min\{1 - 1/p, o_{1/\psi}\} = \gamma$$

then

$$\mathrm{comp}(\varepsilon, \rho, \psi) = \Theta\left(\varepsilon^{-1/s}\left(\int_0^{c_1 L^{-1}(c_2\varepsilon)} \kappa^{1/\gamma}(t)\, dt\right)^{\gamma/s}\right)$$

for some positive constants c_1 and c_2.

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